

# ON SOFIC MONOIDS

TULLIO CECCHERINI-SILBERSTEIN AND MICHEL COORNAERT

**ABSTRACT.** We investigate the notion of soficity for monoids. A group is sofic as a group if and only if it is sofic as a monoid. All finite monoids, all commutative monoids, all free monoids, all cancellative one-sided amenable monoids, all multiplicative monoids of matrices over a field, and all monoids obtained by adjoining an identity element to a semigroup without identity element are sofic. On the other hand, although the question of the existence of a non-sofic group remains open, we prove that the bicyclic monoid is not sofic. This shows that there exist finitely presented amenable inverse monoids that are non-sofic.

## 1. INTRODUCTION

Sofic groups were introduced at the end of the last century by M. Gromov [12] and B. Weiss [19]. The class of groups they constitute is very large since it includes in particular all locally residually amenable groups and hence all linear groups. Actually, the question whether or not every group is sofic remains open up to now although several experts in the field think that the answer to this question should be negative. Roughly speaking, a group is sofic when it can be well approximated by finite symmetric groups. Sofic groups satisfy certain finiteness properties that are important in the theory of dynamical systems and operator algebras. For example, it is known that every sofic group is surjunctive [19], hyperlinear [9], and has stably finite group algebras whatever the ground field [8]. For an introduction to the theory of sofic groups, the reader is referred to the excellent survey paper [17] or to [4, Chapter 7].

The theme of soficity was fruitfully developed in several other directions: weakly-sofic groups [11], linear sofic groups [1], sofic groupoids of measure-preserving transformations [6], [2], and sofic measure-preserving equivalence relations [7]. In each of these settings, the basic question of the existence of a non-sofic object remains still unanswered.

The goal of the present note is to investigate soficity for monoids, i.e., sets equipped with a binary operation that is associative and admits an identity element. A group is sofic as a monoid if and only if it is sofic as a group. As every submonoid of a sofic monoid is sofic, this implies that every monoid that can be embedded into a sofic group is itself sofic. Consequently, all free monoids, all cancellative one-sided amenable monoids, are sofic. We shall also see that all finite monoids and all commutative monoids are sofic. As there exist finite monoids as well as commutative monoids that are not cancellative, this shows

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in particular that there are sofic monoids that cannot be embedded into groups. On the other hand, we shall prove that the bicyclic monoid is non-sofic. Thus there exist finitely presented amenable inverse monoids that are not sofic.

## 2. BACKGROUND MATERIAL

**2.1. Semigroups and monoids.** A *semigroup* is a set equipped with an associative binary operation. Unless stated otherwise, we will use multiplicative notation for the binary operation on a semigroup.

Let  $S$  be a semigroup.

Given  $s \in S$ , we denote by  $L_s$  and  $R_s$  the left and right multiplication by  $s$ , that is, the maps  $L_s: S \rightarrow S$  and  $R_s: S \rightarrow S$  defined by  $L_s(t) = st$  and  $R_s(t) = ts$  for all  $t \in S$ . An element  $s \in S$  is called *left-cancellable* (resp. *right-cancellable*) if the map  $L_s$  (resp.  $R_s$ ) is injective. One says that an element  $s \in S$  is cancellable if it is both left-cancellable and right-cancellable. The semigroup  $S$  is called *left-cancellative* (resp. *right-cancellative*, resp. *cancellative*) if every element in  $S$  is left-cancellable (resp. right-cancellable, resp. cancellable).

Given semigroups  $S_1$  and  $S_2$ , a map  $\varphi: S_1 \rightarrow S_2$  is called a *semigroup morphism* if it satisfies  $\varphi(st) = \varphi(s)\varphi(t)$  for all  $s, t \in S_1$ .

Let  $\mathcal{P}$  be a property of semigroups (e.g., being finite). One says that a semigroup  $S$  is *locally  $\mathcal{P}$*  if every finitely generated subsemigroup of  $S$  satisfies  $\mathcal{P}$ . One says that a semigroup  $S$  is *residually  $\mathcal{P}$*  if, given any pair of distinct elements  $s_1, s_2 \in S$ , there exists a semigroup  $T$  satisfying  $\mathcal{P}$  and a semigroup morphism  $\varphi: S \rightarrow T$  such that  $\varphi(s_1) \neq \varphi(s_2)$ . One says that a semigroup  $S$  is *locally embeddable* into the class of semigroups satisfying  $\mathcal{P}$  (or, for short, *locally embeddable into  $\mathcal{P}$* ) if, for every finite subset  $K \subset S$ , there exists a semigroup  $T$  satisfying  $\mathcal{P}$  and a map  $\varphi: S \rightarrow T$  satisfying the following properties: (1) the restriction of  $\varphi$  to  $K$  is injective, (2) for all  $k_1, k_2 \in K$ , one has  $\varphi(k_1 k_2) = \varphi(k_1)\varphi(k_2)$  (note that  $\varphi$  is not required to be globally injective nor to be a semigroup morphism).

**Proposition 2.1.** *Let  $\mathcal{P}$  be a property of semigroups. Suppose that any finite product of semigroups satisfying  $\mathcal{P}$  also satisfies  $\mathcal{P}$ . Then every locally residually  $\mathcal{P}$  semigroup is locally embeddable into  $\mathcal{P}$ .*

*Proof.* Suppose that  $S$  is a locally residually  $\mathcal{P}$  semigroup and  $K \subset S$  is a finite subset. Denote by  $T$  the semigroup generated by  $K$ . Let  $D := \{\{s, t\} : s, t \in K \text{ and } s \neq t\}$ . As  $T$  is residually  $\mathcal{P}$ , for each  $d = \{s, t\} \in D$ , there exist a semigroup  $R_d$  satisfying  $\mathcal{P}$  with a semigroup morphism  $\psi_d: T \rightarrow R_d$  such that  $\psi_d(s) \neq \psi_d(t)$ . By our hypothesis, the product semigroup  $P := \prod_{d \in D} R_d$  satisfies  $\mathcal{P}$ . The product semigroup morphism  $\psi := \prod_{d \in D} \psi_d: T \rightarrow P$  is injective on  $K$ . By extending arbitrarily  $\psi$  to  $S$ , we get a map  $\varphi: S \rightarrow P$  that is injective on  $K$  and satisfies  $\varphi(k_1 k_2) = \varphi(k_1)\varphi(k_2)$  for all  $k_1, k_2 \in K$ . This shows that  $S$  is locally embeddable into  $\mathcal{P}$ .  $\square$

A semigroup that is locally embeddable into the class of finite semigroups is called an *LEF-semigroup*. As a product of finitely many finite semigroups is finite, we deduce from Proposition 2.1 the following:

**Corollary 2.2.** *Every locally residually finite semigroup is an LEF-semigroup. In particular, every residually finite semigroup and every locally finite semigroup is an LEF-semigroup.*  $\square$

A semigroup  $S$  is called an *inverse semigroup* if, for every  $s \in S$ , there exists a unique element  $x \in S$  such that  $s = sxs$  and  $x = xsx$ .

A *monoid* is a semigroup admitting an identity element. If  $M$  is a monoid, we denote its identity element by  $1_M$ .

Given two monoids  $M_1$  and  $M_2$ , a semigroup morphism  $\varphi: M_1 \rightarrow M_2$  is called a *monoid morphism* if it satisfies  $\varphi(1_{M_1}) = 1_{M_2}$ . A *submonoid* of a monoid  $M$  is a subsemigroup  $N \subset M$  such that  $1_M \in N$ .

**2.2. Symmetric monoids and the Hamming metric.** Let  $X$  be a set. We denote by  $\text{Map}(X)$  the symmetric monoid of  $X$ , i.e., the set consisting of all maps  $f: X \rightarrow X$  with the composition of maps as the monoid operation. The identity element of the symmetric monoid  $\text{Map}(X)$  is the identity map  $\text{Id}_X: X \rightarrow X$ .

Suppose that  $X$  is a non-empty finite set. The *Hamming metric*  $d_X^{\text{Ham}}$  on  $\text{Map}(X)$  is the metric defined by

$$d_X^{\text{Ham}}(f, g) = \frac{1}{|X|} |\{x \in X : f(x) \neq g(x)\}|$$

for all  $f, g \in \text{Map}(X)$  (we use  $|\cdot|$  to denote cardinality of finite sets). Note that  $0 \leq d_X^{\text{Ham}}(f, g) \leq 1$  for all  $f, g \in \text{Map}(X)$ .

Suppose now that  $X_1, X_2, \dots, X_n$  is a finite sequence of non-empty finite sets. Consider the Cartesian product  $X = \prod_{1 \leq i \leq n} X_i$  and the natural semigroup morphism  $\Phi: \prod_{1 \leq i \leq n} \text{Map}(X_i) \rightarrow \text{Map}(X)$  given by

$$\Phi(f)(x) = (f_1(x_1), \dots, f_n(x_n))$$

for all  $f = (f_i)_{1 \leq i \leq n} \in \prod_{1 \leq i \leq n} \text{Map}(X_i)$  and  $x = (x_i)_{1 \leq i \leq n} \in X$ .

**Proposition 2.3.** *With the above notation, one has*

$$(2.1) \quad d_X^{\text{Ham}}(\Phi(f), \Phi(g)) = 1 - \prod_{1 \leq i \leq n} (1 - d_{X_i}^{\text{Ham}}(f_i, g_i))$$

for all  $f = (f_i)_{1 \leq i \leq n}$  and  $g = (g_i)_{1 \leq i \leq n}$  in  $\prod_{1 \leq i \leq n} \text{Map}(X_i)$ .

*Proof.* The formula immediately follows from the equality

$$\{x \in X : \Phi(f)(x) = \Phi(g)(x)\} = \prod_{1 \leq i \leq n} \{x_i \in X_i : f_i(x_i) = g_i(x_i)\}$$

after taking cardinalities of both sides.  $\square$

### 3. SOFIC MONOIDS

**Definition 3.1.** Let  $M$  be a monoid,  $K \subset M$  and  $\varepsilon, \alpha > 0$ . Let  $N$  be a monoid equipped with a metric  $d$ .

A map  $\varphi: M \rightarrow N$  is called a  $(K, \varepsilon)$ -*morphism* if it satisfies

$$d(\varphi(k_1 k_2), \varphi(k_1) \varphi(k_2)) \leq \varepsilon \quad \text{for all } k_1, k_2 \in K$$

and

$$d(\varphi(1_M), 1_N) \leq \varepsilon.$$

A map  $\varphi: M \rightarrow N$  is said to be  $(K, \alpha)$ -*injective* if it satisfies

$$d(\varphi(k_1), \varphi(k_2)) \geq \alpha$$

for all distinct  $k_1, k_2 \in K$ .

If  $X$  is a non-empty finite set, we equip its symmetric monoid  $\text{Map}(X)$  with its Hamming metric.

**Definition 3.2.** A monoid  $M$  is called *sofic* if it satisfies the following condition: for every finite subset  $K \subset M$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $X$  and a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ .

**Proposition 3.3.** *Let  $M$  be a monoid. Then the following conditions are equivalent:*

- (a)  $M$  is sofic;
- (b) for every  $0 < \alpha < 1$ , for every finite subset  $K \subset M$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $X$  and a  $(K, \alpha)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ .
- (c) there exists  $0 < \alpha < 1$  such that, for every finite subset  $K \subset M$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $X$  and a  $(K, \alpha)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ .

*Proof.* Let  $0 < \alpha < 1$ ,  $K \subset M$  a finite subset and  $\varepsilon > 0$ . Choose  $\varepsilon' > 0$  small enough so that  $\alpha \leq 1 - \varepsilon'$  and  $\varepsilon' \leq \varepsilon$ . If  $M$  is sofic, we can find a non-empty finite set  $X$  and a  $1 - \varepsilon'$ -injective  $\varepsilon'$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ . Then  $\varphi$  is a  $\alpha$ -injective  $\varepsilon$ -morphism. This shows that (a) implies (b).

Condition (b) trivially implies (c).

To complete the proof, it suffices to show that (c) implies (a). We use the technique of “amplification” (see for example [17, Theorem 3.5], [11, Proposition 3.4]). Suppose that (c) is satisfied for some  $0 < \alpha < 1$ . Let  $K \subset M$  be a finite subset and  $\varepsilon > 0$ . Choose an integer  $n \geq 1$  large enough so that

$$(3.1) \quad 1 - (1 - \alpha)^n \geq 1 - \varepsilon$$

and then  $\varepsilon' > 0$  such that

$$(3.2) \quad 1 - (1 - \varepsilon')^n \leq \varepsilon.$$

By (c), there exist a non-empty finite set  $X$  and a map  $\varphi: M \rightarrow \text{Map}(X)$  that is a  $(K, \alpha)$ -injective  $(K, \varepsilon')$ -morphism.

Consider the diagonal monoid morphism  $\Delta: \text{Map}(X) \rightarrow \text{Map}(X^n)$  defined by

$$\Delta(f)(x_1, \dots, x_n) := (f(x_1), \dots, f(x_n))$$

for all  $f \in \text{Map}(X)$  and  $(x_1, \dots, x_n) \in X^n$ . Then the composite map  $\psi := \Delta \circ \varphi: M \rightarrow \text{Map}(X^n)$  satisfies, for all distinct  $k_1, k_2 \in K$ ,

$$\begin{aligned} d_{X^n}^{\text{Ham}}(\psi(k_1), \psi(k_2)) &= d_{X^n}^{\text{Ham}}(\Delta(\varphi(k_1)), \Delta(\varphi(k_2))) \\ &= 1 - (1 - d_X^{\text{Ham}}(\varphi(k_1), \varphi(k_2)))^n && \text{(by (2.1))} \\ &\geq 1 - (1 - \alpha)^n \\ &\geq 1 - \varepsilon && \text{(by (3.1)).} \end{aligned}$$

On the other hand, for all  $k_1, k_2 \in K$ ,

$$\begin{aligned} d_{X^n}^{\text{Ham}}(\psi(k_1 k_2), \psi(k_1)\psi(k_2)) &= d_{X^n}^{\text{Ham}}(\Delta(\varphi(k_1 k_2)), \Delta(\varphi(k_1))\Delta(\varphi(k_2))) \\ &= d_{X^n}^{\text{Ham}}(\Delta(\varphi(k_1 k_2)), \Delta(\varphi(k_1)\varphi(k_2))) \\ &= 1 - (1 - d_X^{\text{Ham}}(\varphi(k_1 k_2), \varphi(k_1)\varphi(k_2)))^n && \text{(by (2.1))} \\ &\leq 1 - (1 - \varepsilon')^n \\ &\leq \varepsilon && \text{(by (3.2)).} \end{aligned}$$

Moreover, we also have

$$\begin{aligned} d_{X^n}^{\text{Ham}}(\psi(1_M), \text{Id}_{X^n}) &= d_{X^n}^{\text{Ham}}(\Delta(\varphi(1_M)), \text{Id}_{X^n}) \\ &= d_{X^n}^{\text{Ham}}(\Delta(\varphi(1_M)), \Delta(\text{Id}_X)) \\ &= 1 - (1 - d_X^{\text{Ham}}(\varphi(1_M), \text{Id}_X))^n && \text{(by (2.1))} \\ &\leq 1 - (1 - \varepsilon')^n \\ &\leq \varepsilon && \text{(by (3.2)).} \end{aligned}$$

We deduce that the map  $\psi: M \rightarrow \text{Map}(X^n)$  is a  $(K, \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism. This shows that (c) implies (a).  $\square$

**Proposition 3.4.** *Let  $G$  be a group. Then  $G$  is sofic as a group if and only if it is sofic as a monoid.*

*Proof.* The fact that any group that is sofic as a monoid is also sofic as a group is clear if we compare our Definition 3.1 and Definition 3.2 above with Definition 1.1 and Definition 1.2 in [10]. The converse implication, namely that any group that is sofic as a group is also sofic as a monoid, follows from our definitions and Lemma 2.1 in [10].  $\square$

**Proposition 3.5.** *Every submonoid of a sofic monoid is sofic.*

*Proof.* Let  $M$  be a sofic monoid and  $N$  a submonoid of  $M$ . Fix a finite subset  $K \subset N$  and  $\varepsilon > 0$ . As  $M$  is sofic, there exists a non-empty finite set  $X$  and a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ . Then the restriction map  $\varphi|_N: N \rightarrow \text{Map}(X)$  is a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism. This shows that the monoid  $N$  is sofic.  $\square$

**Proposition 3.6.** *Every locally sofic monoid is sofic.*

*Proof.* Let  $M$  be a locally sofic monoid. Let  $K \subset M$  be a finite subset and  $\varepsilon > 0$ . Denote by  $N$  the submonoid of  $M$  generated by  $K$ . As  $N$  is sofic, there exist a non-empty finite set  $X$  and a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism  $\psi: M \rightarrow \text{Map}(X)$ . By extending arbitrarily  $\psi$  to  $M$ , we get a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ . This shows that  $M$  is sofic.  $\square$

**Proposition 3.7.** *Let  $(M_i)_{i \in I}$  be a family of sofic monoids. Then the product monoid  $M := \prod_{i \in I} M_i$  is also sofic.*

*Proof.* For each  $i \in I$ , let  $\pi_i: M \rightarrow M_i$  denote the projection morphism. Fix a finite subset  $K \subset M$  and  $\varepsilon > 0$ . Then there exists a finite subset  $J \subset I$  such that the projection  $\pi_J: M \rightarrow M_J := \prod_{j \in J} M_j$  is injective on  $K$ . Choose a constant  $0 < \eta < 1$  small enough so that

$$(3.3) \quad 1 - (1 - \eta)^{|J|} \leq \varepsilon$$

and  $\eta \leq \varepsilon$ .

Since the monoid  $M_j$  is sofic for each  $j \in J$ , there exist a nonempty finite set  $X_j$  and a  $(\pi_j(K), 1 - \eta)$ -injective  $(\pi_j(K), \eta)$ -morphism  $\varphi_j: M_j \rightarrow \text{Map}(X_j)$ . Consider the nonempty finite set  $X := \prod_{j \in J} X_j$  and the map  $\varphi: M \rightarrow \text{Map}(X)$  defined by

$$\varphi(m)(x) := (\varphi_j(m_j)(x_j))_{j \in J}$$

for all  $m = (m_i)_{i \in I} \in M$  and  $x = (x_j)_{j \in J} \in X$ . For all  $k = (k_i)_{i \in I}, k' = (k'_i)_{i \in I} \in K$ , we have

$$\begin{aligned} d_X^{\text{Ham}}(\varphi(kk'), \varphi(k)\varphi(k')) &= 1 - \prod_{j \in J} \left(1 - d_{X_j}^{\text{Ham}}(\varphi_j(k_j k'_j), \varphi_j(k_j)\varphi_j(k'_j))\right) \quad (\text{by (2.1)}) \\ &\leq 1 - (1 - \eta)^{|J|} \\ &\leq \varepsilon \quad (\text{by (3.3)}). \end{aligned}$$

We also have

$$\begin{aligned} d_X^{\text{Ham}}(\varphi(1_m), \text{Id}_X) &= 1 - \prod_{j \in J} \left(1 - d_{X_j}^{\text{Ham}}(\varphi_j(1_{M_j}), \text{Id}_{X_j})\right) \quad (\text{by (2.1)}) \\ &\leq 1 - (1 - \eta)^{|J|} \\ &\leq \varepsilon \quad (\text{by (3.3)}). \end{aligned}$$

On the other hand, if  $k$  and  $k'$  are distinct elements in  $K$ , then there exists  $j_0 \in J$  such that  $k_{j_0} \neq k'_{j_0}$ . This implies

$$\begin{aligned} d_X^{\text{Ham}}(\varphi(k), \varphi(k')) &= 1 - \prod_{j \in J} (1 - d_{X_j}^{\text{Ham}}(\varphi_j(k_j), \varphi_j(k'_j))) \quad (\text{by (2.1)}) \\ &\geq 1 - (1 - d_{X_{j_0}}^{\text{Ham}}(\varphi_{j_0}(k_{j_0}), \varphi_{j_0}(k'_{j_0}))) \\ &\geq 1 - \eta \quad (\text{since } \varphi_{j_0} \text{ is } (K_{j_0}, 1 - \eta)\text{-injective}) \\ &\geq 1 - \varepsilon \quad (\text{since } \eta \leq \varepsilon). \end{aligned}$$

It follows that  $\varphi$  is a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism. This shows that  $M$  is sofic.  $\square$

**Corollary 3.8.** *Let  $(M_i)_{i \in I}$  be a family of sofic monoids. Then their direct sum  $M = \bigoplus_{i \in I} M_i$  is also sofic.*

*Proof.* This immediately follows from Proposition 3.5 and Proposition 3.7 since  $M = \bigoplus_{i \in I} M_i$  is a submonoid of the product monoid  $\prod_{i \in I} M_i$ .  $\square$

**Corollary 3.9.** *If a monoid  $M$  is the limit of a projective system of sofic monoids then  $M$  is sofic.*

*Proof.* If  $M$  is the limit of a projective system of monoids  $(M_i)_{i \in I}$  then  $M$  is a submonoid of the product  $\prod_{i \in I} M_i$ . Thus, it follows from Proposition 3.5 and Proposition 3.7 that  $M$  is sofic if every  $M_i$ ,  $i \in I$ , is sofic.  $\square$

**Proposition 3.10.** *Every monoid that is locally embeddable into the class of sofic monoids is itself sofic.*

*Proof.* Let  $M$  be a monoid that is locally embeddable into the class of sofic monoids. Let  $K \subset S$  be a finite subset and  $\varepsilon > 0$ . We want to show that there exist a non-empty finite set  $X$  and a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism  $\varphi: M \rightarrow \text{Map}(X)$ . Without loss of generality, we may assume  $1_M \in K$ . By definition of local embeddability, there exist a sofic monoid  $S$  and a map  $\psi: M \rightarrow S$  which is injective on  $K$  and satisfies

$$(3.4) \quad \psi(k_1 k_2) = \psi(k_1) \psi(k_2) \quad \text{for all } k_1, k_2 \in K.$$

Let  $K' := \psi(K)$  and let  $N$  denote the submonoid of  $S$  generated by  $K'$ . Note that (3.4) implies that  $\psi(1_M) = 1_N = 1_S$ . Since  $N$  is sofic by Proposition 3.5, there exist a non-empty finite set  $X$  and a  $(K', 1 - \varepsilon)$ -injective  $(K', \varepsilon)$ -morphism  $\varphi': N \rightarrow \text{Map}(X)$ . Then the composite map  $\varphi' \circ \psi: M \rightarrow \text{Map}(X)$  is clearly a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism. This shows that  $M$  is a sofic monoid.  $\square$

**Corollary 3.11.** *Every locally residually sofic monoid is sofic. In particular, every residually sofic monoid and every locally sofic monoid is sofic.*

*Proof.* This immediately follows from Proposition 2.1, Proposition 3.7, and Proposition 3.10.  $\square$

#### 4. EXAMPLES OF SOFIC MONOIDS

**Proposition 4.1.** *Every finite monoid is sofic.*

*Proof.* Any monoid  $M$  is isomorphic to a submonoid of the symmetric monoid  $\text{Map}(M)$  via the Cayley map  $m \mapsto L_m$  that sends every  $m \in M$  to the left multiplication by  $m$ . As every submonoid of a sofic monoid is itself sofic by Proposition 3.5, it suffices to prove that the symmetric monoid of any finite set is sofic.

Let  $X$  be a finite set of cardinality  $|X| \geq 1$  and let  $\alpha := 1/|X|$ . Then, for every  $\varepsilon > 0$  and every  $K \subset \text{Map}(X)$ , the identity morphism  $\text{Id}_{\text{Map}(X)}: \text{Map}(X) \rightarrow \text{Map}(X)$  is a  $(K, \alpha)$ -injective  $(K, \varepsilon)$ -morphism. Thus, the monoid  $\text{Map}(X)$  satisfies condition (c) in Proposition 3.3. This shows that  $\text{Map}(X)$  is sofic.  $\square$

From Proposition 4.1, Proposition 3.10, and Corollary 2.2, we deduce the following result.

**Corollary 4.2.** *Every LEF-monoid is sofic. In particular, every locally residually finite monoid, and hence every residually finite monoid and every locally finite monoid, is sofic.*  $\square$

**Proposition 4.3.** *Every commutative monoid is sofic.*

*Proof.* This follows from Corollary 4.2, since, by a result of Mal'cev [16] (see also [13], [3]), every commutative semigroup is locally residually finite.  $\square$

**Corollary 4.4.** *Every free monoid is sofic.*

*Proof.* This follows from Corollary 4.2 since every free monoid is residually finite.  $\square$

**Corollary 4.5.** *Let  $K$  be a field and let  $n \geq 1$  be an integer. Then the multiplicative monoid  $\text{Mat}_n(K)$  formed by all  $n \times n$  matrices with entries in  $K$  is sofic.*

*Proof.* This follows from Corollary 4.2, since, by a result of Mal'cev [15], the multiplicative monoid  $\text{Mat}_n(K)$  is locally residually finite.  $\square$

*Remark.* In notes by Stallings [18], it is shown that the field  $K$  in Mal'cev result can be replaced by any commutative unital ring.

**Proposition 4.6.** *All cancellative one-sided amenable monoids are sofic.*

*Proof.* It is known [20, Corollary 3.6] that every cancellative left-amenable monoid is isomorphic to a submonoid of an amenable group. As the opposite semigroup of a right-amenable semigroup is left-amenable and every group is isomorphic to its opposite, we deduce that every cancellative right-amenable semigroup is also isomorphic to a submonoid of an amenable group. Thus, the result follows from Proposition 3.5 and the fact that every amenable group is sofic as a group (see for instance [4, Proposition 7.5.6]) and hence sofic as a monoid by Proposition 3.4.  $\square$

**Proposition 4.7.** *Let  $S$  be a semigroup that is not a monoid. Let  $M$  denote the monoid obtained from  $S$  by adjoining an identity element. Then  $M$  is sofic.*

*Proof.* By definition,  $S$  is a subsemigroup of  $M = S \cup \{1_M\}$  and  $1_M \notin S$ . Let  $K$  be a finite subset of  $M$  and  $\varepsilon > 0$ . Let  $Y := K \cup K^2$  denote the subset of  $M$  consisting of all elements that are in  $K$  or may be written as the product of two elements in  $K$ . Choose an arbitrary element  $y_0 \notin Y$  and a finite set  $Z$  disjoint from  $Y \cup \{y_0\}$ . Let  $X := Y \cup \{y_0\} \cup Z$  and consider the map  $\varphi: M \rightarrow \text{Map}(X)$  defined as follows. We take  $\varphi(1_M) = \text{Id}_X$  and, for  $s \in S$ , define  $\varphi(s) \in \text{Map}(X)$  by

$$\varphi(s)x = \begin{cases} s & \text{if } s \in Y \text{ and } x \in Z \\ sx & \text{if } s \in Y, x \in Y, \text{ and } sx \in Y \\ y_0 & \text{otherwise} \end{cases}$$

for all  $x \in X$ .

For  $k_1, k_2 \in K \setminus \{1_M\}$ , we have  $k_1 k_2 \in Y \setminus \{1_M\}$  so that

$$(\varphi(k_1)\varphi(k_2))(z) = \varphi(k_1)(k_2) = k_1 k_2 = \varphi(k_1 k_2)(z).$$



As  $\varphi(k_1k_2) = \varphi(k_1)\varphi(k_2)$  if  $k_1 = 1_M$  or  $k_2 = 1_M$ , we deduce that

$$d_X^{\text{Ham}}(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)) \leq 1 - \frac{|Z|}{|X|} \quad \text{for all } k_1, k_2 \in K.$$

On the other hand, if  $z \in Z$ , we have  $\varphi(1_M)(z) = z$  and  $\varphi(k)(z) = k$  for all  $k \in K \setminus \{1_M\}$ . It follows that

$$d_X^{\text{Ham}}(\varphi(k_1), \varphi(k_2)) \geq \frac{|Z|}{|X|} \quad \text{for all distinct } k_1, k_2 \in K.$$

Consequently,  $\varphi$  is a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism for  $|Z|$  large enough. This shows that the monoid  $M$  is sofic.  $\square$

*Remark.* One may rephrase Proposition 4.7 by saying that every monoid  $M$  in which the equation  $xy = 1_M$  implies  $x = y = 1_M$  is sofic.

## 5. NON-SOFICITY OF THE BICYCLIC MONOID

The *bicyclic monoid* is the monoid  $B$  given by the presentation  $B = \langle p, q : pq = 1 \rangle$ . Every element  $s \in B$  may be uniquely written in the form  $s = q^a p^b$ , where  $a = a(s)$  and  $b = b(s)$  are non-negative integers. The bicyclic monoid may also be viewed as a submonoid of the symmetric monoid  $\text{Map}(\mathbb{N})$  of the set of non-negative integers by regarding  $p$  and  $q$  as the maps respectively defined by

$$p(n) = \begin{cases} n-1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases} \quad \text{and} \quad q(n) = n+1 \quad \text{for all } n \in \mathbb{N}.$$

**Theorem 5.1.** *The bicyclic monoid  $B = \langle p, q : pq = 1 \rangle$  is not sofic.*

Let us first establish the following result.

**Lemma 5.2.** *Let  $X$  be a non-empty finite set and let  $\varepsilon > 0$ . Let  $f, g \in \text{Map}(X)$  and suppose that  $d_X^{\text{Ham}}(fg, \text{Id}_X) \leq \varepsilon$ . Then one has  $d_X^{\text{Ham}}(gf, \text{Id}_X) \leq 2\varepsilon$ .*

*Proof.* Since  $d_X^{\text{Ham}}(fg, \text{Id}_X) \leq \varepsilon$ , the set  $X_1 := \{x \in X : fg(x) = x\}$  satisfies  $|X_1| \geq (1 - \varepsilon)|X|$ . As the restriction of  $g$  to  $X_1$  is injective, we have that  $|g(X_1)| = |X_1| \geq (1 - \varepsilon)|X|$ . It follows that the set  $X_0 := X_1 \cap g(X_1)$  satisfies  $|X_0| \geq (1 - 2\varepsilon)|X|$ . Let now  $x \in X_0$  and let us denote by  $y$  the unique element in  $X_1$  such that  $x = g(y)$ . Then we have  $gf(x) = gfg(y) = g(y) = x$ . We deduce that  $d_X^{\text{Ham}}(fg, \text{Id}_X) \leq 1 - |X_0|/|X| \leq 2\varepsilon$ .  $\square$

*Proof of Theorem 5.1.* Let  $K := \{1, p, q, qp\}$  and  $0 < \varepsilon < \frac{1}{7}$ . Suppose that  $X$  is a non-empty finite set and that  $\varphi : B \rightarrow \text{Map}(X)$  is a  $(K, 1 - \varepsilon)$ -injective  $(K, \varepsilon)$ -morphism.

Consider the maps  $f := \varphi(p)$  and  $g := \varphi(q)$ . We then have

$$\begin{aligned}
 d_X^{\text{Ham}}(fg, \text{Id}_X) &= d_X^{\text{Ham}}(\varphi(p)\varphi(q), \text{Id}_X) \\
 &\leq d_X^{\text{Ham}}(\varphi(pq), \text{Id}_X) + d_X^{\text{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad (\text{by the triangle inequality}) \\
 &= d_X^{\text{Ham}}(\varphi(1_B), \text{Id}_X) + d_X^{\text{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad (\text{since } pq = 1_B) \\
 &\leq 2\varepsilon \quad (\text{since } \varphi \text{ is a } (K, \varepsilon)\text{-morphism}).
 \end{aligned}$$

Applying Lemma 5.2, we obtain

$$(5.1) \quad d_X^{\text{Ham}}(gf, \text{Id}_X) \leq 4\varepsilon.$$

Finally, using the triangle inequality, we get

$$\begin{aligned}
 d_X^{\text{Ham}}(\varphi(qp), \varphi(1_B)) &\leq d_X^{\text{Ham}}(\varphi(qp), gf) + d_X^{\text{Ham}}(gf, \text{Id}_X) + d_X^{\text{Ham}}(\varphi(1_B), \text{Id}_X) \\
 &\leq d_X^{\text{Ham}}(\varphi(qp), \varphi(q)\varphi(p)) + 4\varepsilon + d_X^{\text{Ham}}(\varphi(1_B), \text{Id}_X) \quad (\text{by (5.1)}) \\
 &\leq 6\varepsilon \quad (\text{since } \varphi \text{ is a } (K, \varepsilon)\text{-morphism}).
 \end{aligned}$$

This contradicts the fact that  $\varphi$  is  $(K, 1 - \varepsilon)$ -injective since  $qp$  and  $1_B$  are distinct elements of  $K$  and  $6\varepsilon < 1 - \varepsilon$ . Consequently, the monoid  $B$  is not sofic.  $\square$

**Corollary 5.3.** *There exist finitely presented amenable inverse monoids that are not sofic.*

*Proof.* It is known that the bicyclic monoid is an amenable inverse monoid (see for example [5, Example 2, page 311]).  $\square$

*Remark.* It follows from Corollary 5.3 that we cannot remove the cancellativity hypothesis in Proposition 4.6.

**Corollary 5.4.** *Every monoid containing a submonoid isomorphic to the bicyclic monoid is non-sofic.*

*Proof.* This immediately follows from Proposition 3.5 and Theorem 5.1.  $\square$

**Corollary 5.5.** *Let  $X$  be an infinite set. Then the symmetric monoid  $\text{Map}(X)$  is not sofic.*

*Proof.* This follows from Corollary 5.4 since  $\text{Map}(X)$  contains a submonoid isomorphic to  $\text{Map}(\mathbb{N})$  and hence a submonoid isomorphic to the bicyclic monoid.  $\square$

**Corollary 5.6.** *Let  $K$  be a field and let  $E$  be an infinite-dimensional vector space over  $K$ . Let  $\mathcal{L}(E)$  denote the monoid consisting of all endomorphisms of  $E$  with the composition of maps as the monoid operation. Then  $\mathcal{L}(E)$  is not sofic.*

*Proof.* Let  $(e_x)_{x \in X}$  be a basis of  $E$ . Then the symmetric monoid  $\text{Map}(X)$  embeds into  $\mathcal{L}(E)$  via the map that sends each  $f \in \text{Map}(X)$  to the unique endomorphism  $u$  of  $E$  such that  $u(e_x) = e_{f(x)}$  for all  $x \in X$ . Thus  $\mathcal{L}(E)$  is not sofic by Corollary 5.5.  $\square$

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DIPARTIMENTO DI INGEGNERIA, UNIVERSITÀ DEL SANNIO, C.SO GARIBALDI 107, 82100 BENEVENTO, ITALY

*E-mail address:* tceccher@mat.uniroma3.it

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UMR 7501, UNIVERSITÉ DE STRASBOURG ET CNRS, 7 RUE RENÉ-DESCARTES, 67000 STRASBOURG, FRANCE

*E-mail address:* coornaert@math.unistra.fr